Force Transmission Analyses with Dimensionally Homogeneous Jacobian Matrices for Parallel Manipulators

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To avoid the unit inconsistency problem in the conventional Jacobian matrix, new formulation of a dimensionally homogeneous inverse Jacobian matrix for parallel manipulators with a planar mobile platform by using three end-effector points was presented (Kim and Ryu, 2003). This paper presents force relationships between joint forces and Cartesian forces at the three End-Effector points. The derived force relationships can then be used for analyses of the input/output force transmission. These analyses, forward and inverse force transmission analyses, depend on the singular values of the derived unit consistent Jacobian matrix. Using the proposed force relationship, a numerical example is presented for actuator size design of a 3-RRR planar parallel manipulator.

Key Words: Jacobian, Force Transmission Analysis, Parallel Manipulator

Nomenclature

- T_j : Three distinct and noncollinear End-Effector (EE) points ($j=1, 2, 3$)
- O : Center of the base reference frame
- C : Center of the mobile frame
- x, y, z : Axes of the base reference frame
- x', y', z' : Axes of the mobile reference frame attached to the mobile platform
- A_i : Center of the base universal joint of leg i
- B_i : Center of the platform universal (or spherical) joint of leg i
- $\mathbf{q}$: The vector defined by the coordinates of three EE points describing the motion of the mobile platform (e.g. $\mathbf{q}=[x_1, y_1, z_1, x_2, y_2, z_2, x_3, y_3, z_3]^T$)
- $k_{i,j}$: The coefficients that are functions of the geometry of the mobile platform joints B_i and the pre-selected three points T_j
- λ_i : The magnitude of the actuating length
- $\mathbf{n}_i$: The unit vector of articular coordinate i
- $\dot{\mathbf{t}}$: Time differentiation of three EE with respect to the fixed world coordinate system
- Λ : Vector of articular coordinates, $\Lambda=[\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6]^T$
- $\mathbf{J}$: Jacobian matrix of the manipulator
- $l_{i,1}, l_{i,2}$: The link lengths
- $\sigma_{\min}, \sigma_{\max}$: The minimum and maximum singular values of the Jacobian matrix
- $\mathbf{F}_1, \mathbf{F}_2, \mathbf{F}_3$: The external forces acting on three points
- τ : The joint forces
- $\boldsymbol{\Phi}_q$: The constraint Jacobian matrix
- $\dot{\mathbf{x}}_C$: The twist of end effector (C)
- $\mathbf{v}_C, \boldsymbol{\omega}$: The linear velocity and angular velocity of the center of the mobile

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frame

$\mathbf{J}_q$: (9×6) transformation matrix that is mapping between the twist of end effector and the Cartesian velocity of three EE points

$\mathbf{J}_A$, $\mathbf{J}_x$: The conventional non-homogeneous (6×6) inverse and forward Jacobian matrices

$\mathbf{F}_{ext}$: The external Cartesian forces at three EE point

$\mathbf{R}$: Rotation matrix mapping the coordinates from the base frame to the mobile frame

1. Introduction

In order to avoid unit inconsistency problem in the conventional Jacobian matrix for parallel manipulators, Kim and Ryu (2003) proposed a new inverse Jacobian formulation based on the three End-Effector (EE) point coordinate. The derivation was based on a velocity relationship between actuator joint space and Cartesian space that is composed of three EE point coordinates. However, a question arises: can this new Jacobian be used to describe the force relationship between the joint and Cartesian spaces? This paper answers the question by presenting the force relationship between actuator joint forces and Cartesian forces at three EE points.

Tsai (1999) and Asada (1986) had derived force relationship between joint forces and Cartesian forces. In this paper, however, we introduce a different approach. We utilize the coordinates of three different points at the end-effector to characterize the kinematic and force relationship. This gives a solution to the unit inconsistency problem in the conventional Jacobian matrix. Based on this idea we present a new formulation of force relationships between actuator joint forces and Cartesian forces.

When a parallel manipulator executes a given task, such as grinding, grasping, brushing, lifting up, and so on, its end-effector exerts forces and moments on workpiece. These forces and moments are generated by actuators of the parallel mechanism in the joint space. Hence, finding force relationship between task and joint spaces is a practical and basic requirement in the design and control of robot manipulators. The force relationship can then be used for analysis of the input/output force transmission. These kinetostatic performance analyses can provide essential information (Kosuge et al., 1993 ; Kim and Choi, 1999, 2001 ; Choi, 2003) such as how much task forces can be produced by applied actuator forces. They also provide a basis for structural design of the links and bearings of a robot manipulator and for selection of appropriate size of actuators. For physically meaningful force relationship, however, unit consistency of Jacobian matrix is necessary, since the force transmission analysis depends on the singular values or condition number of $\mathbf{JJ}^T$ (as discussed in Section 3) (Kim and Choi, 1999, 2001 ; Doty et al., 1993, 1995).

This paper is organized as follows ; Section 2 describes inverse and forward force relationships between the joint and the Cartesian spaces at three EE points based on the dimensionally homogeneous Jacobian matrices. The next section presents force transmission analyses with the derived force relationships. Section 4 illustrates a numerical example to select actuators based on the previous inverse force transmission analysis method. Conclusions are presented in the last section.

2. Force Relationships Between Joint and Cartesian Spaces

2.1 Inverse force relationship

Consider a general 6-6 parallel manipulator with a planar mobile platform as shown in Fig. 1. Here, the platform joints $B_i(i=1, 2, \cdots, 6)$ are assumed on the same moving plane while the base joints denoted by A_i are not necessarily on a plane. Let $\mathbf{q}$ be the vector defined by the coordinates of three EE points describing the motion of the mobile platform :

$$\mathbf{q}=[x_1, y_1, z_1, x_2, y_2, z_2, x_3, y_3, z_3]^T \quad (1)$$

Since B_i and T_j points are on the same plane of a mobile platform, the coordinates of the

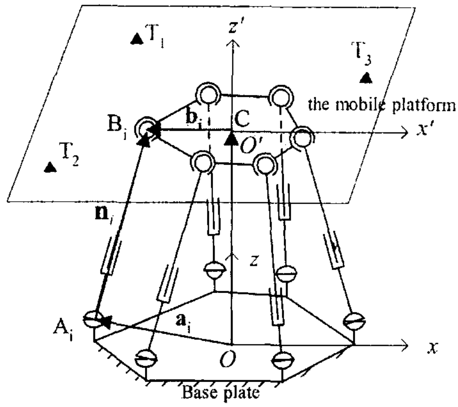

Fig. 1 6-6 General parallel manipulator (GPM)

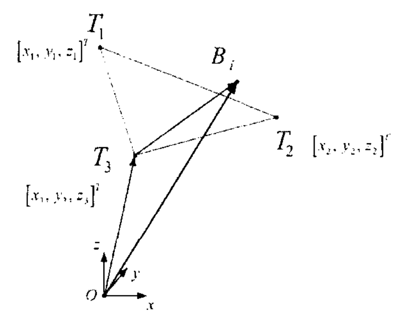

Fig. 2 Representing the coordinates of the platform joints in the coordinates of the three EE points

platform joints B_i in the absolute coordinate frame can easily be expressed in terms of the coordinates of the three EE points (Fig. 2) as

$$\mathbf{OB}_i=\begin{bmatrix} k_{i,1}x_1+k_{i,2}x_2+k_{i,3}x_3 \\ k_{i,1}y_1+k_{i,2}y_2+k_{i,3}y_3 \\ k_{i,1}z_1+k_{i,2}z_2+k_{i,3}z_3 \end{bmatrix},\ i=1,\ \cdots,\ 6 \quad (2)$$

where $k_{i,j}(i=1,\ 2,\ \cdots,\ 6\ ;\ j=1,\ 2,\ 3)$ are dimensionless constants and $k_{i,1}+k_{i,2}+k_{i,3}=1$. Indeed, this is true because

$$\mathbf{OB}_i=\mathbf{OT}_3+k_{i,1}\mathbf{T}_3\mathbf{T}_1+k_{i,2}\mathbf{T}_3\mathbf{T}_2 \quad (3)$$

Note that if all of the platform joints are not coplanar with the three points, the expression in Eq. (2) will be more complicated and the following derivation should be changed substantially.

The coefficients $k_{i,j}$ in Eq. (3) are functions of the geometry of the mobile platform joints B_i and the pre-selected three points T_j. If the global vectors are transformed to the local moving reference frame, Eq. (3) can be written as

$$\mathbf{B}'_i=k_{i,1}\mathbf{T}'_1+k_{i,2}\mathbf{T}'_2+(1-k_{i,1}-k_{i,2})\mathbf{T}'_3 \quad (4)$$

where $\mathbf{B}'_i$ and $\mathbf{T}'_j$ points are (2×1) constant vectors with x' and y' coordinates in the reference frame fixed on the mobile platform. Rewriting Eq. (4) gives

$$\mathbf{B}'_i-\mathbf{T}'_3=k_{i,1}(\mathbf{T}'_1-\mathbf{T}'_3)+k_{i,2}(\mathbf{T}'_2-T'_3) \quad i=1,\ 2,\ \cdots,\ 6 \quad (5)$$

Then, for each i, the two unknowns ($k_{i,1}$ and $k_{i,2}$) in Eq. (5) can be obtained in terms of constant $\mathbf{B}'_i$ and $\mathbf{T}'_j$ coordinates as long as the three EE points are distinct and noncollinear. The practical choice of three points, however, is governed in part by numerical conditioning of Eq. (5). Since equilateral triangular layout of three points with the triangle center being at the geometric center of B_i points generates good numerical conditions, it is recommended for the optimal design of an axi-symmetrical mobile platform (Kim and Ryu, 2003).

The new Jacobian matrix by using the three EE points can be derived as follows: First, consider the 6-dof Gough-Stewart parallel manipulator which has six translational actuators. The inverse kinematic relationship from the motion of the moving platform to the actuator lengths can easily be derived as

$$\begin{aligned} A_iB_i=\lambda_i\mathbf{n}_i&=k_{i,1}\mathbf{OT}_1+k_{i,2}\mathbf{OT}_2+k_{i,3}\mathbf{OT}_3-\mathbf{OA}_i \\ &=k_{i,1}\mathbf{t}_1+k_{i,2}\mathbf{t}_2+k_{i,3}\mathbf{t}_3-\mathbf{a}_i \end{aligned} \quad (6)$$

where λ_i is the magnitude of the actuating length and $\mathbf{n}_i$ is a unit vector. Time differentiation of Eq. (6) with respect to the fixed world coordinate system gives

$$\dot{\lambda}_i\mathbf{n}_i+\lambda_i\dot{\mathbf{n}}_i=k_{i,1}\dot{\mathbf{t}}+k_{i,2}\dot{\mathbf{t}}_2+k_{i,3}\dot{\mathbf{t}}_3 \quad (7)$$

where $\dot{\mathbf{t}}=[\dot{x}_1,\ \dot{y}_i,\ \dot{z}_i]^T$.

Since $\mathbf{n}_i$ is a unit vector, $\mathbf{n}_i^T\mathbf{n}_i=1$ and $\mathbf{n}_i^T\dot{\mathbf{n}}_i=0$. Therefore, multiplication of $\mathbf{n}_i^T$ with Eq. (7) gives

$$\dot{\lambda}_i=k_{i,1}\mathbf{n}_i^T\dot{\mathbf{t}}_1+k_{i,2}\mathbf{n}_i^T\dot{\mathbf{t}}_2+k_{i,3}\mathbf{n}_i^T\dot{\mathbf{t}}_3 \quad (8)$$

The velocity relationship between actuator joint space and Cartesian space that is composed of three EE point coordinates can then be expressed as (Kim and Ryu, 2003)

$$\dot{\Lambda}=\mathbf{J}\dot{\mathbf{q}} \tag{9}$$

where

$$\dot{\Lambda}=[\dot{\lambda}_1,\ \dot{\lambda}_2,\ \dot{\lambda}_3,\ \dot{\lambda}_4,\ \dot{\lambda}_5,\ \dot{\lambda}_6]^T$$

and

$$\dot{\mathbf{q}}=[\dot{x}_1,\ \dot{y}_1,\ \dot{z}_1,\ \dot{x}_2,\ \dot{y}_2,\ \dot{z}_2,\ \dot{x}_3,\ \dot{y}_3,\ \dot{z}_3]^T$$

This inverse Jacobian matrix $\mathbf{J}$ is an actual Jacobian, i.e., a matrix of partial derivatives of Cartesian coordinates with respect to the joint variables. If the three EE points are on the plane in which B_i points are located, the matrix $\mathbf{J}$ then can be compactly given as

$$\mathbf{J}=\begin{bmatrix} k_{1,1}\mathbf{n}_1^T & k_{1,2}\mathbf{n}_1^T & k_{1,3}\mathbf{n}_1^T \\ k_{2,1}\mathbf{n}_2^T & k_{2,2}\mathbf{n}_2^T & k_{2,3}\mathbf{n}_2^T \\ k_{3,1}\mathbf{n}_3^T & k_{3,2}\mathbf{n}_3^T & k_{3,3}\mathbf{n}_3^T \\ k_{4,1}\mathbf{n}_4^T & k_{4,2}\mathbf{n}_4^T & k_{4,3}\mathbf{n}_4^T \\ k_{5,1}\mathbf{n}_5^T & k_{5,2}\mathbf{n}_5^T & k_{5,3}\mathbf{n}_5^T \\ k_{6,1}\mathbf{n}_6^T & k_{6,2}\mathbf{n}_6^T & k_{6,3}\mathbf{n}_6^T \end{bmatrix} \tag{10}$$

where $\mathbf{n}_i$ denotes the unit vectors along vector A_iB_i and $k_{i,j}$ are constants. Note that all elements in the new (6×9) Jacobian matrix are dimensionless because $k_{i,j}$ and unit vectors are dimensionless. Note also that it can be shown that we can have a dimensionally homogeneous Jacobian matrix even if the three EE points are not coplanar with B_i points. In this general spatial parallel manipulators case, however, more complicated derivation is necessary (e.g., Eqs. (2) and (5) should be modified), as mentioned by Kim and Ryu (2003).

In order to derive a new force relationship between actuator joint forces and Cartesian forces at three EE points, it is assumed that every force and moment on the mobile platform is decomposed into point forces ($\mathbf{F}_1$, $\mathbf{F}_2$, $\mathbf{F}_3$) at three points (T_1, T_2, T_3) that may be considered as three grasping points or three connecting joints to the mobile platform (say; B_1, B_3, B_5) as shown in Fig. 3. Note that the decomposition

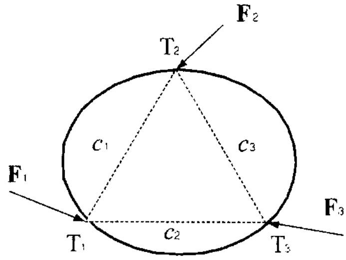

Fig. 3 External forces at three EE points on mobile platform

is not unique. In the conventional force relationship, joint space forces are mapped into Cartesian space forces as a combination of three translational forces at the origin of mobile platform reference frame and three torques about the local reference frame axes. In this case, the Jacobian loses its dimensional homogeneity. From Eq. (9) the virtual displacement relationship can be written as

$$\delta\boldsymbol{\Lambda}=\mathbf{J}\delta\mathbf{q} \tag{11}$$

Then, virtual work principle can be stated as

$$\delta \mathrm{W}=\tau^T\delta\boldsymbol{\Lambda}-\mathbf{F}^T\delta\mathbf{q}=0 \tag{12}$$

where the force vector $\mathbf{F}$ includes every internal or external force that is applied equivalently at three EE points.

Inserting Eq. (11) into Eq. (12) gives

$$(\boldsymbol{\tau}^T\mathbf{J}-\mathbf{F}^T)\,\delta\mathbf{q}=0 \tag{13}$$

The elements in the virtual displacement vector $\delta\mathbf{q}$ are not independent due to the following distance constraints:

$$\Phi_i=(\mathbf{T}_i-\mathbf{T}_j)^T(\mathbf{T}_i-\mathbf{T}_j)-c_i^2=0 \tag{14}$$

for $(i,\ j)=(1,\ 2),\ (2,\ 3),\ (3,\ 1)$

where c_i's are the constant distances between T_i and T_j points.

Therefore,

$$\delta\boldsymbol{\Phi}=\boldsymbol{\Phi}_q\delta\mathbf{q}=0 \tag{15}$$

where $\boldsymbol{\Phi}_q$ can be expressed as

$$\boldsymbol{\Phi}_q^T=\begin{bmatrix} (x_1-x_2) & 0 & -(x_3-x_1) \\ (y_1-y_2) & 0 & -(y_3-y_1) \\ (z_1-z_2) & 0 & -(z_3-z_1) \\ -(x_1-x_2) & (x_2-x_3) & 0 \\ -(y_1-y_2) & (y_2-y_3) & 0 \\ -(z_1-z_2) & (z_2-z_3) & 0 \\ 0 & -(x_2-x_3) & (x_3-x_1) \\ 0 & -(y_2-y_3) & (y_3-y_1) \\ 0 & -(z_2-z_3) & (z_3-z_1) \end{bmatrix} \tag{16}$$

From Eqs. (13) and (15), the Lagrangian multiplier theorem (Haug, 1989) states that

$$(\boldsymbol{\tau}^T\mathbf{J}-\mathbf{F}^T+\boldsymbol{\alpha}^T\boldsymbol{\Phi}_q)\,\delta\mathbf{q}=0 \tag{17}$$

where $\boldsymbol{\alpha}$ is the (3×1) Lagrangian multiplier vector that can be physically interpreted as the constraint reaction forces among the three rigid points on the mobile platform.

Since Eq. (17) is true for any arbitrary $\delta\mathbf{q}$ vector, the Cartesian forces are represented as

$$\mathbf{F}=\mathbf{J}^T\boldsymbol{\tau}+\boldsymbol{\Phi}_q^T\boldsymbol{\alpha} \tag{18}$$

From Eq. (16), the second term in Eq. (18) can be rewritten as

$$\boldsymbol{\Phi}_q^T\boldsymbol{\alpha}=\begin{bmatrix} \alpha_1(x_1-x_2)-\alpha_3(x_3-x_1) \\ \alpha_1(y_1-y_2)-\alpha_3(y_3-y_1) \\ \alpha_1(z_1-z_2)-\alpha_3(z_3-z_1) \\ -\alpha_1(x_1-x_2)-\alpha_2(x_2-x_3) \\ -\alpha_1(y_1-y_2)-\alpha_2(y_2-y_3) \\ -\alpha_1(z_1-z_2)-\alpha_2(z_2-z_3) \\ -\alpha_2(x_2-x_3)-\alpha_3(x_3-x_1) \\ -\alpha_2(y_2-y_3)-\alpha_3(y_3-y_1) \\ -\alpha_2(z_2-z_3)-\alpha_3(z_3-z_1) \end{bmatrix} = \begin{bmatrix} \alpha_1\mathbf{f}_{12}+\alpha_3\mathbf{f}_{13} \\ \alpha_2\mathbf{f}_{23}+\alpha_1\mathbf{f}_{21} \\ \alpha_3\mathbf{f}_{31}+\alpha_2\mathbf{f}_{32} \end{bmatrix} \tag{19}$$

where the force vector $\mathbf{f}_{ij}$ acts along the T_iT_j line as shown in Fig. 4. Therefore, these forces are in a single plane and are self-equilibrated (self-canceled).

The fact that the term $\boldsymbol{\Phi}_q^T\alpha$ is a self-canceling internal force vector means that this term has no relationship with the external Cartesian forces at three EE points.

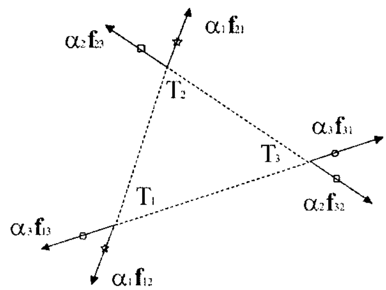

Fig. 4 Self-canceling internal forces at three EE points on mobile platform

Therefore, only the term $\mathbf{J}^T\boldsymbol{\tau}$ is directly related to the external Cartesian forces at three EE points and Eq. (18) can be restated as

$$\mathbf{F}_{ext}=\mathbf{J}^T\boldsymbol{\tau} \tag{20}$$

Since the unit consistent Jacobian matrix is used in Eq. (20), this equation can be used in the optimal design and control of parallel manipulators without any scale-varying problems.

2.2 Forward force relationship

Joint forces may be obtained from Eq. (20) that can be rewritten in a linear equation form as

$$\mathbf{J}^T\boldsymbol{\tau}=\mathbf{F}_{ext} \tag{21}$$

where $\mathbf{J}^T\in\Re^{n\times m}$, $\boldsymbol{\tau}\in\Re^m$, and $\mathbf{F}_{ext}\in\Re^n$. This equation, however, represents an overdetermined system of linear equations. For convenience, this equation can be modified to an underdetermined system using the other "direction" of the mapping (Gosselin, 1992) that is more useful for the forward force transmission problem.

The end effector (C) of a GPM is shown in Fig. 1 where the reference frame $\Re(O-xyz)$ is fixed to the base of the GPM while frame $\Re'(O'-x'y'z')$ is attached to the origin of the mobile plate. The twist of end effector (C) can be defined as

$$\dot{\mathbf{x}}_C=[\mathbf{v}_C^T,\ \boldsymbol{\omega}^T]^T \tag{22}$$

where $\mathbf{v}_C$ is the velocity of the origin of the mobile frame and stands for the angular velocity vector of the platform.

Now, in order to obtain the new transformation Jacobian matrix that is mapping from the twist of end effector (C) to the Cartesian velocity of three EE points $T_j (j=1, 2, 3)$, we should derive the kinematic relationship between vectors $\dot{\mathbf{x}}_C$ and $\dot{\mathbf{q}}$. This can be written as

$$\dot{\mathbf{q}}=\mathbf{J}_q\dot{\mathbf{x}}_C \tag{23}$$

where $\mathbf{J}_q$ is a (9×6) transformation matrix. The position vectors of three EE points with respect to O' will be given by

$$\mathbf{T}'_j=[O'T_j]_{\mathfrak{R}'}=[x'_j, y'_j, z'_j]^T, (j=1, 2, 3) \tag{24}$$

where the prime means that the vector is represented with respect to the body reference frame.

Let the rotation matrix representing the change of coordinates from $\mathfrak{R}'$ to $\mathfrak{R}$ be denoted by $\mathbf{R}$ matrix. The $\mathbf{R}$ matrix can be written as

$$\mathbf{R}=\begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \tag{25}$$

The position vectors of three EE points with respect to O will be given by

$$\mathbf{T}_j=[OT_j]_{\mathfrak{R}}=[x_j, y_j, z_j]^T, (j=1, 2, 3) \tag{26}$$

Therefore, the velocity equations will be given by

$$\begin{aligned}[\dot{\mathbf{t}}_j]_{\mathfrak{R}}&=[\dot{x}_j, \dot{y}_j, \dot{z}_j] \\ &=[\mathbf{v}_C]_{\mathfrak{R}}+[\omega\times\mathbf{T}_j]_{\mathfrak{R}}, (j=1, 2, 3)\end{aligned} \tag{27}$$

which leads to a transformation matrix in Eq. (28)

$$\mathrm{J}_q=\begin{bmatrix} 1\,0\,0 & 0 & (r_{31}x'_1+r_{32}y'_1+r_{33}z'_1) & -(r_{21}x'_1+r_{22}y'_1+r_{23}z'_1) \\ 0\,1\,0 & -(r_{31}x'_1+r_{32}y'_1+r_{33}z'_1) & 0 & (r_{11}x'_1+r_{12}y'_1+r_{13}z'_1) \\ 0\,0\,1 & (r_{21}x'_1+r_{22}y'_1+r_{23}z'_1) & -(r_{11}x'_1+r_{12}y'_1+r_{13}z'_1) & 0 \\ 1\,0\,0 & 0 & (r_{31}x'_2+r_{32}y'_2+r_{33}z'_2) & -(r_{21}x'_2+r_{22}y'_2+r_{23}z'_2) \\ 0\,1\,0 & -(r_{31}x'_2+r_{32}y'_2+r_{33}z'_2) & 0 & (r_{11}x'_2+r_{12}y'_2+r_{13}z'_2) \\ 0\,0\,1 & (r_{21}x'_2+r_{22}y'_2+r_{23}z'_2) & -(r_{11}x'_2+r_{12}y'_2+r_{13}z'_2) & 0 \\ 1\,0\,0 & 0 & (r_{31}x'_3+r_{32}y'_3+r_{33}z'_3) & -(r_{21}x'_3+r_{22}y'_3+r_{23}z'_3) \\ 0\,1\,0 & -(r_{31}x'_3+r_{32}y'_3+r_{33}z'_3) & 0 & (r_{11}x'_3+r_{12}y'_3+r_{13}z'_3) \\ 0\,0\,1 & (r_{21}x'_3+r_{22}y'_3+r_{33}z'_3) & -(r_{11}x'_3+r_{12}y'_3+r_{13}z'_3) & 0 \end{bmatrix} \tag{28}$$

Then, by using the notation of (Gosselin, 1990), the standard velocity equations of the parallel manipulator can be written as

$$\mathbf{J}_A\dot{\Lambda}=\mathbf{J}_x\dot{\mathbf{x}}_C \tag{29}$$

where $\mathbf{J}_A$ and J_x are the conventional non-homogeneous (6×6) inverse and forward Jacobian matrices.

These matrices are expressed as

$$\mathbf{J}_x=\begin{bmatrix} \mathbf{n}_1^T & (\mathbf{b}_1\times\mathbf{n}_1)^T \\ \mathbf{n}_2^T & (\mathbf{b}_2\times\mathbf{n}_2)^T \\ \vdots & \vdots \\ \mathbf{n}_6^T & (\mathbf{b}_6\times\mathbf{n}_6)^T \end{bmatrix}, \text{ and} \tag{30}$$

$$\mathbf{J}_A=\mathbf{E}_{6\times6} (6\times6 \text{ identity matrix})$$

where $\mathbf{n}_i$ and $\mathbf{b}_i$ denote the unit vectors along vector A_iB_i and vector CB_i, respectively.

The latter equation can also be written as

$$\dot{\mathbf{x}}_C=\mathbf{J}_x^{-1}\mathbf{J}_A\dot{\Lambda} \tag{31}$$

Then, by premultiplying Eq. (23) by matrix $\mathbf{J}_q$, it becomes

$$\dot{\mathbf{q}}=\mathbf{J}_q\mathbf{J}_x^{-1}\mathbf{J}_A\dot{\Lambda}=\mathbf{J}_3\dot{\Lambda} \tag{32}$$

where matrix $\mathbf{J}_3$ is then a (9×6) matrix, the dimensional homogeneity of which can be verified by the MAPLE Software. Note that this forward Jacobian matrix can not be defined for the singular configurations that can be manifested by $\mathbf{J}_x^{-1}$

The virtual work principle can be stated for Eq. (32) as

$$\delta W=\boldsymbol{\tau}^T\delta\Lambda-\mathbf{F}^T\delta\mathbf{q}=\tau^T\delta\Lambda-\mathbf{F}^T\mathbf{J}_3\delta\Lambda=0 \tag{33}$$

Since the components of vector $\delta\Lambda$ are independent, it can be simply written as

$$\boldsymbol{\tau}=\mathbf{J}_3^T\mathbf{F} \tag{34}$$

which is an underdetermined system of linear equations.

3. Force Transmission Analyses

Input/output force (or velocity) transmission capabilities are important in kinetostatic performances of robotic manipulator for design and control. There are two input/output force transmission analyses: the forward force transmission analysis to determine the magnitude bounds of the force vector at three EE points for given magnitude of joint actuator forces or torques and the inverse force transmission analysis to determine the magnitude bounds of joint actuating forces for the given magnitude of the

Cartesian force. This section presents force transmission analyses based on the previously derived dimensionally homogeneous Jacobian matrix.

3.1 Inverse force transmission analysis

The inverse force transmission analysis can provide a basis for sizing links and bearings of a robot manipulator and for selecting appropriate force size of actuators. The inverse force transmission analysis can be formulated as

$$\| \mathbf{F}_{ext} \|^2 = \mathbf{F}_{ext}^T \mathbf{F}_{ext} = \boldsymbol{\tau}^T \mathbf{J}\mathbf{J}^T \boldsymbol{\tau} \tag{35}$$

where $\|\cdot\|$ denotes the Euclidean norm of a vector.

Eq. (35) shows that the actuator joint forces form an hyperellipsoid in the Euclidean space which lies in the directions of eigenvectors of the $\mathbf{J}\mathbf{J}^T$ matrix and the joint force $\| \boldsymbol{\tau} \|$ bounds for the given Cartesian force $\| \mathbf{F}_{ext} \|$ are given by the square roots of the singular values of the $\mathbf{J}\mathbf{J}^T$ matrix :

$$\sigma_{\min} \| \mathbf{F}_{ext} \| \le \| \boldsymbol{\tau} \| \le \sigma_{\max} \| \mathbf{F}_{ext} \| \tag{36}$$

where $\sigma_{\min}$ and $\sigma_{\max}$ stand for the minimum and the maximum singular values of the dimensionally homogeneous $\mathbf{J}$ matrix. If $\| \mathbf{F}_{ext} \|$ is the magnitude of the required Cartesian force, the magnitude of actuator's force should be larger than $\sigma_{\min} \| \mathbf{F}_{ext} \|$. Singular values in Eq. (36) can be computed by the SVD (singular value decomposition) theorem (Kadama and Suda, 1978 ; Yoshikawa, 1985). Note that these results are invariant to changes of units since the used Jacobian is dimensionally homogeneous.

3.2 Forward force transmission analysis

The forward force transmission analysis provides the extreme magnitudes and their directions of the output forces for given joint forces. The magnitude bounds of input joint forces can be given as

$$\| \boldsymbol{\tau} \|^2 = \boldsymbol{\tau}^T \tau \le 1 \tag{37}$$

Finally, the extreme magnitudes and their directions of the output forces for given joint forces can be obtained as

$$\| \boldsymbol{\tau} \|^2 = \boldsymbol{\tau}^T \boldsymbol{\tau} = \mathbf{F}^T \mathbf{J}_3 \mathbf{J}_3^T \mathbf{F} \tag{38}$$

Eq. (38) shows that the Cartesian forces at three EE points on the mobile platform form an hyperellipsoid in the Euclidean space which lies in the directions of eigenvectors of the $\mathbf{J}_3\mathbf{J}_3^T$ matrix. Then the output force bounds for $\| \mathbf{F} \|$ with respect to input force $\| \tau \|$ are given by the square roots of the singular values of the $\mathbf{J}_3\mathbf{J}_3^T$ matrix :

$$\sigma_{3\min} \| \boldsymbol{\tau} \| \le \| \mathbf{F} \| \le \sigma_{3\max} \| \boldsymbol{\tau} \| \tag{39}$$

where $\sigma_{3\min}$ and $\sigma_{3\max}$ stand for the minimum and the $\mathbf{J}_3$ maximum singular values of the matrix. Note that since the force ellipsoid is based on the dimensionally homogeneous Jacobian, the mapping does not change with changes of scale.

4. A Numerical Example of Actuator Size Selection

As an application example of the previous input/output force transmission analyses, this section presents an actuator size selection problem for a simple 3-RRR planar parallel manipulator that is shown in Fig. 5 in which the actuated joints are denoted by A_i and the passive revolute joints at the mobile platform are denoted by B_i. Link lengths are denoted by $l_{i,1}$ and $l_{i,2} (i=1, 2, 3)$ and radii to the joints A_i or B_i from the origin of reference frames are denoted by r_a or r_b, respectively. In this case, we can select three EE points $T_j (j=1, 2, 3)$ as connecting joint points $B_i (i=1, 2, 3)$ and can derive a consistent (3×6) dimensionally homogeneous Jacobian matrix of 3-DOF planar parallel manipulator (Kim and Ryu, 2003).

Now, we select appropriate size of actuators that guarantees force transmission capability of given Cartesian forces at any arbitrary configuration in the entire workspace. In this example, we consider only the constant-orientation workspace shown in Fig. 5.

The constant-orientation workspace (or translation workspace) is defined as the set of locations of the mobile platform center that may be reached when its orientation is fixed (Merlet, 1998a ; Merlet et al., 1998b ; Gosselin, 1996).

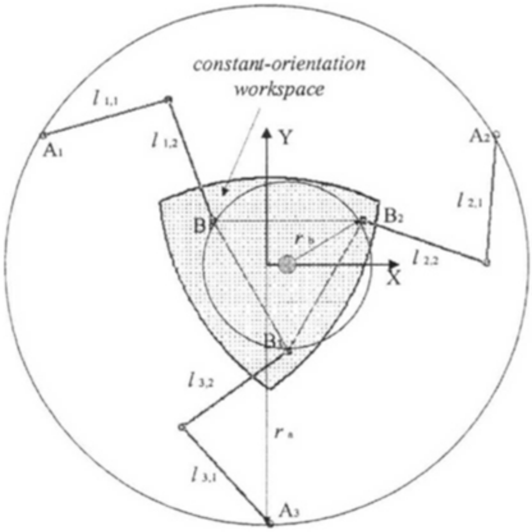

Fig. 5 Constant-orientation workspace for 3-RRR parallel manipulator

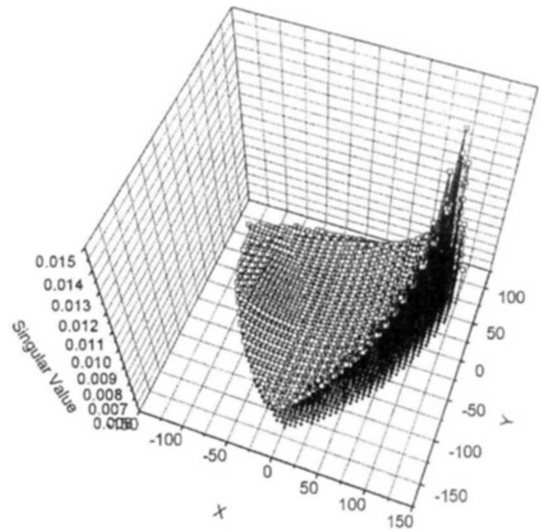

Fig. 6 σ_{min} in whole constant-orientation workspace

Table 1 Dimensions of 3-RRR parallel manipulator

$l_{i,1}$	150 mm
$l_{i,2}$	150 mm
r_a	300 mm
r_b	100 mm

When the unit magnitude of Cartesian force is required, the magnitude of actuator's force should be larger than σ_{min} in Eq. (36) at every point in the translational workspace. Selection of actuator size is then to find the maximum σ_{min} value in the entire workspace. In other words,

$$\| \boldsymbol{\tau} \| \geq \max\{ \sigma_{min}(V) \} \cdot \| \mathbf{F}_{ext} \| \tag{40}$$

where V represents whole translational workspace of the manipulator.

Figure 6 shows σ_{min} values on the entire translational workspace.

Figure 7 shows the configuration of 3-RRR parallel manipulator at the maximum value of $\sigma_{min}(V)$ that occurs at the boundary of the workspace. From this result, we could conclude that the size of actuators should be larger than 0.017 N·m to generate unity magnitude of Cartesian force vector (i.e., $\| \mathbf{F}_{ext} \| = 1$ N).

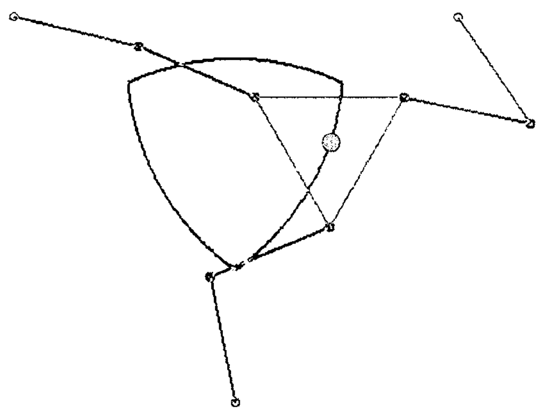

Fig. 7 The manipulator configuration at $\max\{ \sigma_{min}(V) \}$

5. Conclusions

In this paper, we derived the relationship between joint forces of parallel manipulator and Cartesian forces at three EE points on the mobile platform. This derivation is based on the proposed unit consistent Jacobian matrix (Kim and Ryu, 2003). Using this force relationship, we presented input/output force transmission analyses: forward and inverse force transmission analyses. An example of selecting actuator size of 3-RRR planar parallel manipulators has been presented when the unit magnitude of Cartesian force vector is required. Since this force transmission analysis depends on the singular values of the Jacobian matrix, the proposed dimensionally homogeneous Jacobian can be useful for it.

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